On a Generalized Fifth-Order Integrable Evolution Equation and its Hierarchy

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A general form of a fifth-order nonlinear evolution equation is considered. The Helmholtz solution of the inverse variational problem is used to derive conditions under which this equation admits an analytic representation. A Lennard type recursion operator is then employed to construct a hierarchy of Lagrangian equations. It is explicitly demonstrated that the constructed system of equations has a Lax representation and two compatible Hamiltonian structures. The homogeneous balance method is used to derive analytic soliton solutions of the third- and fifth-order equations. – PACS numbers: 47.20.Ky, 42.81.Dp, 02.30.Jr

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1. Introduction

In recent years studies on fifth-order nonlinear evolution equations have received considerable attention, primarily because these equations possess many connections with other integrable equations which play a role in diverse areas of physics, ranging from nonlinear optics [1] to Bose-Einstein condensation [2]. For example, Özer and Döken [3] used a multiple-scale method to derive the fifth-order Korteweg-de Vries (KdV) equation from the higher-order nonlinear Schrödinger equation. On the other hand, a similar method could also be used [4] to obtain the nonlinear Schrödinger equation from fifth-order KdV flow [5], Sawada-Kotera equation [6] and Kaup-Kupershmidt equation [7].

Third-order evolution equations can often be solved either by the use of an inverse spectral method or by taking recourse to a simple change of variables. This is true for both the linear dispersive KdV equation and the nonlinear dispersive Rosenau-Hymann equation [8]. In contrast, it is quite difficult to obtain solutions of the fifth-order equations. This might be another point of interest for recent studies [9] on these equations.

In this work we derive the conditions under which the general fifth-order nonlinear evolution equations

$$u_t = u_{5x} + Auu_{3x} + Bu_xu_{2x} + Cu^2u_x, \quad u = u(x,t) \tag{1}$$

admit an analytic representation [10] or follow from a Lagrangian. Here A, B and C are constant model parameters. The subscripts of u denote partial derivatives with respect to that variable and, in particular, $u_{nx} = \frac{\partial^n u}{\partial x^n}$. We use the fifth-order Lagrangian equation to define an integrable hierarchy. Further, we provide a Lax representation [5] and construct a bi-Hamiltonian structure [11] for the system. The Lagrangian approach to the nonlinear evolution equation has two novel features. First, from the Lagrangians or Lagrangian densities we can construct Hamiltonian densities [12] which form a set of involutive conserved densities of the system. Second, the expression for the Lagrangian represents a useful basis to construct an approximate solution for the evolution equation [13, 14]. We shall, however, use a direct method [15] to obtain explicit analytic soliton solutions.

In Section 2 we deal with the inverse variational problem for (1) and derive relations between the model parameters for the equation to be Lagrangian. We then make use of an appropriate pseudo-differential operator to construct a hierarchy of equations and present results for the first few members of the hierarchy. In Section 3 we find their Lax representations and examine the bi-Hamiltonian structure. The results presented are expected to serve as a useful test of integrability. In Section 4 we present explicit solitonic solutions by using the homogeneous balance method (HB). In Section 5 we present some concluding remarks.

2. Lagrangian System of Equations

In the calculus of variations one is concerned with two types of problems, namely the direct and the inverse problem of Newtonian mechanics. The direct problem is essentially the conventional one in which one first assigns a Lagrangian and then computes the equations of motion through Lagrange equations. As opposed to this, the inverse problem begins with the equation of motion and then constructs a Lagrangian consistent with the variational principle [10]. The inverse problem of the calculus of variation was solved by Helmholtz [16] at the end of the nineteenth century. For continuum mechanics, the Helmholtz version of the inverse problem proceeds by considering an r-tuple of differentiable functions, written as

$$P[v] = P\left(x, v^{(n)}\right) \varepsilon \mathscr{A}^r, \tag{2}$$

and then defining the so-called Fréchet derivative. The Fréchet derivative of P is the differential operator $D_P : \mathscr{A}^q \to \mathscr{A}^r$ and is given by

$$D_P(Q) = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} P[v + \varepsilon Q[v]] \tag{3}$$

for any $Q \in \mathscr{A}^q$. The Helmholtz condition asserts that P is the Euler-Lagrange expression for some variational problem if D_P is self-adjoint. When self-adjointness is guaranteed, a Lagrangian density for P can be explicitly constructed using the homotopy formula

$$\mathscr{L}[v] = \int_0^1 vP[\lambda v]\mathrm{d}\lambda. \tag{4}$$

In the following we shall demand the Helmholtz condition to be valid for (1). This will provide us with certain constraints between the model parameters of (1) to follow from a Lagrangian density.

A single evolution equation $u_t = P[u], u \in \mathbb{R}$ is never the Euler-Lagrange equation of a variational problem [16]. One common trick to put a single evolution equation into a variational form is to replace u by a potential function:

$$u = -w_x,\ w = w(x,t). \tag{5}$$

The function w is often called the Casimir potential. In terms of the Casimir potential, (1) reads

$$w_{xt} = P[w_x], \tag{6}$$

where

$$\begin{aligned} P[w_x] = {} & w_{6x} - Aw_x w_{4x} - Bw_{2x}w_{3x} \\ & + C{w_x}^2 w_{2x}. \end{aligned} \tag{7}$$

From (3) and (7) we obtain

$$\begin{aligned} D_P = {} & D_{6x} - Aw_x D_{4x} - Aw_{4x}D_x - Bw_{2x}D_{3x} \\ & - Bw_{3x}D_{2x} + C{w_x}^2 D_{2x} + 2Cw_x w_{2x} D_x. \end{aligned} \tag{8}$$

To construct the adjoint operator D_P^* of the above Fréchet derivative we rewrite (8) as

$$D_P = \sum_j P_j[w_x]D_j \tag{9}$$

and make use of the definition [16]

$$D_P^* = \sum (-D)_j \cdot P_j, \tag{10}$$

meaning that for any $Q \varepsilon \mathscr{A}$

$$D_P^* Q = \sum_j (-D)_j [P_j Q]. \tag{11}$$

This gives

$$\begin{aligned} D_P^* = {} & D_{6x} - Aw_x D_{4x} - (3A - B)w_{4x}D_x \\ & - (4A - B)w_{2x}D_{3x} - (6A - 2B)w_{3x}D_{2x} \\ & + Cw_x^2 D_{2x} + 2Cw_x w_{2x} D_x. \end{aligned} \tag{12}$$

Demanding variational self-adjointness we obtain from (8) and (12)

$$B = 2A, \tag{13}$$

while C remains unrestricted. Thus the nonlinear equation

$$u_t = u_{5x} + Auu_{3x} + 2Au_x u_{2x} + Cu^2 u_x \tag{14}$$

forms a Lagrangian system. We note that the Lax equation [5] with $A = 10$, $B = 20$ and $C = 30$ and the Ito equation [17] with $A = 3$, $B = 6$ and $C = 2$ are of the form (14), while the Sawada-Kotera equation with $A = B = C = 5$ and the Kaup-Kupershmidt equation with $A = 10$, $B = 25$ and $C = 20$ are non-Lagrangian.

We now use the fifth-order Lagrangian equation (14) to define an integrable hierarchy. To that end we introduce a pseudo-differential or integro-differential operator Λ which acts on a generic function $f(x)$ to give [18]

$$\Lambda f(x) = f_{xx} - puf(x) + qu_x \int_x^{+\infty} \mathrm{d}y f(y). \quad (15)$$

Further, we introduce a function $g_x^{(n)}$ to follow from

$$\Lambda^n u_x(x,t) = g_x^{(n)}, n = 0,1,2\dots. \quad (16)$$

Here $g_x^{(n)}$ is a polynomial in u and its x-derivatives (up to derivative of order $2n$). Using $f(x) = u_x(x,t)$ in (15), we have

$$\Lambda f(x) = \left(u_{2x} - \frac{p+q}{2}u^2\right)_x. \quad (17)$$

From (16) and (17)

$$\begin{aligned}\Lambda^2 u_x(x,t) = {}& u_{5x} - (2p+q)uu_{3x} \\ & - (3p+4q)u_x u_{2x} \\ & + (p+q)\left(p+\frac{q}{2}\right)u^2 u_x.\end{aligned} \quad (18)$$

Comparing (14) and (18) and identifying $\Lambda^2 u_x(x,t)$ as u_t, we can express p and q in terms of A. This allows us to write

$$C = (p+q)\left(p+\frac{q}{2}\right) = \frac{3A^2}{10}. \quad (19)$$

Therefore, the general form of the fifth-order Lagrangian equation generated by Λ via (16) has the form

$$u_t = u_{5x} + Auu_{3x} + 2Au_x u_{2x} + \frac{3A^2}{10}u^2 u_x. \quad (20)$$

We have used (16) to generate a hierarchy of nonlinear evolution equations for $n = 0,1,2,3$ etc. The first member of the hierarchy ($n = 0$) is a linear equation given by

$$u_t = u_x, \quad (21)$$

while the second one ($n = 1$) is a third-order nonlinear equation

$$u_t = u_{3x} + \frac{3A}{5}uu_x. \quad (22)$$

The third member $(n = 2)$ is obviously the fifth-order equation given in (20). The corresponding seventh- and ninth-order equations are given by

$$\begin{aligned}u_t = {}& u_{7x} + \frac{7A}{5}u_{5x}u + \frac{21A}{5}u_{4x}u_x \\ & + 7Au_{3x}u_{2x} + \frac{7A^2}{10}u^2 u_{3x} \\ & + \frac{14A^2}{5}uu_x u_{2x} + \frac{7A^2}{10}{u_x}^3 + \frac{7A^3}{50}u^3 u_x\end{aligned} \quad (23)$$

and

$$\begin{aligned}u_t = {}& u_{9x} + \frac{9A}{5}u_{7x}u + \frac{36A}{5}u_{6x}u_x \\ & + \frac{84A}{5}u_{5x}u_{2x} + \frac{126A}{5}u_{4x}u_{3x} \\ & + \frac{651A^2}{50}u_x u_{2x}^2 + \frac{483A^2}{50}u_x^2 u_{3x} \\ & + \frac{63A^2}{5}uu_{2x}u_{3x} + \frac{378A^2}{50}uu_x u_{4x} \\ & + \frac{63A^2}{50}u^2 u_{5x} + \frac{63A^3}{50}uu_x^3 \\ & + \frac{126A^3}{50}u^2 u_x u_{2x} + \frac{21A^3}{50}u^3 u_{3x} \\ & + \frac{63A^4}{1000}u^4 u_x.\end{aligned} \quad (24)$$

3. Lax Representation and bi-Hamiltonian Structure

Integrable nonlinear evolution equations admit zero curvature or Lax representation [5]. These equations are characterized by an infinite number of conserved densities which are in involution. Moreover, each number of the hierarchy has a bi-Hamiltonian structure [11]. In the following we demonstrate these three important features for our equations in (20)–(24).

The Lax representation of nonlinear evolution equations is based on the algebra of differential operators. Here one considers two linear operators L and M. The eigenvalue equation for the operator L is given by

$$L\psi = \lambda\psi, \quad (25)$$

where ψ is the eigenfunction and λ is the corresponding eigenvalue. The operator M characterizes the change of eigenfunctions with the parameter t which, in a nonlinear evolution equation, usually corresponds to the time. The general form of this equation is

$$\psi_t = M\psi. \quad (26)$$

If we now invoke the basic result of the inverse spectral method that $\frac{d\lambda}{dt} = 0$ for non-zero eigenfunctions [19], then (25) and (26) will immediately give

$$\frac{\partial L}{\partial t} = [M, L]. \tag{27}$$

Equation (27) is called the Lax equation, and L and M are called the Lax pairs. In the context of Lax's method it is often said that L defines the original spectral problem, while M represents an auxiliary spectral problem. For a given nonlinear evolution equation one needs to find these operators. This is not always a straightforward task. In fact, no systematic procedure has been derived to determine whether a nonlinear partial differential equation can be represented in the form (27).

We shall now find the Lax representation for the hierarchy of equations given in (20) – (24). We first note that, as one goes along the hierarchy, the original spectral problem remains invariant, while the auxiliary spectral problem goes on changing. Keeping this in mind, we take

$$L = \partial_x^2 + \frac{A}{10} u. \tag{28}$$

In writing (28) we have exploited the similarity between (22) and the KdV equation. As regards the auxiliary spectral problem, we postulate that for an evolution equation of the form $u_t = K[u]$ the terms in the Fréchet derivative of $K[u]$ contribute additively with unequal weights to form the operator M such that L and M via (22) reproduces $K[u]$. Of course, there should not be any inconsistency in determining the values of the weight factors. For (22) the Fréchet derivative of $K[u]$ can be obtained as

$$D_P = \partial_x^3 + \frac{3A}{5}(u\partial_x + u_x). \tag{29}$$

We shall, therefore, write

$$M_3 = a\partial_x^3 + \frac{3A}{5}\left(bu\partial_x + cu_x\right). \tag{30}$$

Here the subscript 3 of M indicates that (30) represents the second Lax operator for the third-order equation. We shall follow this convention throughout. Equations (22), (27), (28) and (30) can be combined to get $a = 4$, $b = 1$ and $c = \frac{1}{2}$. Thus we have

$$M_3 = 4\partial_x^3 + \frac{3A}{5}(u\partial_x + \frac{1}{2}u_x). \tag{31}$$

Similarly, we find the results

$$M_5 = 16\partial_x^5 + 4Au\partial_x^3 + 6Au_x\partial_x^2 + 5Au_{2x}\partial_x + \frac{3A^2}{10}u^2\partial_x + \frac{3A}{2}u_{3x} + \frac{3A^2}{10}uu_x, \tag{32}$$

$$\begin{aligned} M_7 = {} & 64\partial_x^7 + \frac{112A}{5}u\partial_x^5 + 56Au_x\partial_x^4 + 84Au_{2x}\partial_x^3 + \frac{14A^2}{5}u^2\partial_x^3 + 70Au_{3x}\partial_x^2 + \frac{42A^2}{5}uu_x\partial_x^2 \\ & + \frac{161A}{5}u_{4x}\partial_x + 7A^2uu_{2x}\partial_x + \frac{147A^2}{30}u_x^2\partial_x + \frac{7A^3}{50}u^3\partial_x + \frac{63A}{10}u_{5x} + \frac{21A^2}{10}uu_{3x} \\ & + \frac{21A^2}{5}u_xu_{2x} + \frac{21A^3}{100}u^2u_x, \end{aligned} \tag{33}$$

and

$$\begin{aligned} M_9 = {} & 256\partial_x^9 + \frac{576A}{5}u\partial_x^7 + \frac{51A}{2}u_{7x} + \frac{2016A}{5}u_x\partial_x^6 + \frac{897A}{5}u_{6x}\partial_x + \frac{4368A}{5}u_{2x}\partial_x^5 + \frac{2814A}{5}u_{5x}\partial_x^2 \\ & + 1176Au_{3x}\partial_x^4 + \frac{5124A}{5}u_{4x}\partial_x^3 + 252A^2u_xu_{2x}\partial_x^2 + \frac{5061A^2}{50}u_{2x}^2\partial_x + \frac{546A^2}{5}u_x^2\partial_x^3 + \frac{3654A^2}{25}u_xu_{3x}\partial_x \\ & + \frac{756A^2}{5}uu_{2x}\partial_x^3 + 126A^2uu_{3x}\partial_x^2 + \frac{609A^2}{10}u_{2x}u_{3x} + \frac{504A^2}{25}u^2\partial_x^5 + \frac{567A^2}{10}uu_{5x} + \frac{516A^2}{5}uu_x\partial_x^4 \\ & + \frac{2967A^2}{50}uu_{4x}\partial_x + \frac{1743A^2}{50}u_xu_{4x} + \frac{189A^3}{100}u_x^3 + \frac{441A^3}{50}uu_x^2\partial_x + \frac{42A^3}{25}u^3\partial_x^3 + \frac{21A^3}{100}u^2u_{3x} \\ & + \frac{378A^3}{50}u^2u_x\partial_x^2 + \frac{63A^3}{10}u^2u_{2x}\partial_x + \frac{189A^3}{25}uu_xu_{2x} + \frac{63A^4}{1000}u^4\partial_x + \frac{63A^4}{500}u^3u_x. \end{aligned} \tag{34}$$

Zakharov and Faddeev [20] developed the Hamiltonian approach to integrability of nonlinear evolution equations in one spatial and one temporal (1+1)-dimension and, in particular, Gardner [21] interpreted the KdV equation as a completely integrable Hamiltonian system with ∂_x as the relevant Hamiltonian operator. A significant development in the Hamiltonian theory is due to Magri [11], who realized that integrable Hamiltonian systems have an additional structure. They are bi-Hamiltonian, i. e. they are Hamiltonian with respect to two different compatible Hamiltonian operators. The bi-Hamiltonian structure of the integrable equation is based on a mathematical formulation that does not make explicit reference to the Lagrangian of the equations in the hierarchy [22]. Here we shall demonstrate that the bi-Hamiltonian structure of the system of equations (20) – (24) can be realized in terms of a set of Hamiltonian densities obtained from the Lagrangians. Using (4), we can obtain the Lagrangian densities for our equations. In particular, we have

$$\mathscr{L}_1 = \frac{1}{2}w_t w_x - \frac{1}{2}w_x^2, \tag{35}$$

$$\mathscr{L}_3 = \frac{1}{2}w_t w_x - \frac{1}{2}w_x w_{3x} + \frac{A}{10}w_x^3, \tag{36}$$

$$\begin{aligned}\mathscr{L}_5 = {} & \frac{1}{2}w_t w_x - \frac{1}{2}w_x w_{5x} + \frac{A}{3}w_x^2 w_{3x} \\ & + \frac{A}{6}w_x w_{2x}^2 - \frac{A^2}{40}w_x^4,\end{aligned} \tag{37}$$

$$\begin{aligned}\mathscr{L}_7 = {} & \frac{1}{2}w_t w_x - \frac{1}{2}w_x w_{7x} + \frac{7A}{10}w_x w_{3x}^2 \\ & - \frac{7A^2}{40}w_x^2 w_{2x}^2 - \frac{7A^2}{40}w_x^3 w_{3x} + \frac{7A^3}{1000}w_x^5,\end{aligned} \tag{38}$$

and

$$\begin{aligned}\mathscr{L}_9 = {} & \frac{1}{2}w_t w_x - \frac{1}{2}w_x w_{9x} - \frac{3A}{5}w_{2x}^2 w_{5x} \\ & + \frac{8A}{5}w_{3x}^3 - \frac{9A}{10}w_x w_{4x}^2 + \frac{7A^2}{40}w_{2x}^4 \\ & + \frac{63A^2}{200}w_x^2 w_{3x}^2 - \frac{21A^3}{100}w_x^3 w_{2x}^2 - \frac{21A^4}{10000}w_x^6.\end{aligned} \tag{39}$$

In the above, $\mathscr{L}_1$ is the Lagrangian density for the linear equation in (21). The other subscripts on $\mathscr{L}$ are self-explanatory. The corresponding Hamiltonian densities are given by

$$\mathscr{H}_1 = \frac{1}{2}u^2, \tag{40}$$

$$\mathscr{H}_3 = \frac{1}{2}uu_{2x} + \frac{A}{10}u^3, \tag{41}$$

$$\mathscr{H}_5 = \frac{1}{2}uu_{4x} + \frac{A}{3}u^2 u_{2x} + \frac{A}{6}uu_x^2 + \frac{A^2}{40}u^4, \tag{42}$$

$$\begin{aligned}\mathscr{H}_7 = {} & \tfrac{1}{2}uu_{6x} + \tfrac{7A}{10}uu_{2x}^2 + \tfrac{7A^2}{40}u^2 u_x^2 \\ & + \frac{7A^2}{40}u^3 u_{2x} + \frac{7A^3}{1000}u^5,\end{aligned} \tag{43}$$

and

$$\begin{aligned}\mathscr{H}_9 = {} & \frac{1}{2}uu_{8x} - \frac{3A}{5}u_x^2 u_{4x} + \frac{8A}{5}u_{2x}^3 \\ & - \frac{9A}{10}uu_{3x}^2 - \frac{63A^2}{200}u^2 u_{2x}^2 - \frac{7A^2}{40}u_x^4 \\ & - \frac{21A^3}{100}u^3 u_x^2 + \frac{21A^4}{10000}u^6.\end{aligned} \tag{44}$$

In the theory of Zakharov and Faddeev [20] and of Gardner [21] the Hamiltonian form of an integrable nonlinear evolution equation reads

$$u_t = \partial_x\left(\frac{\delta\mathscr{H}}{\delta u}\right), \tag{45}$$

where $\mathscr{H}$ is the Hamiltonian densities of that equation. Here $\frac{\delta}{\delta u}$ denotes the usual variational derivative written as

$$\frac{\delta}{\delta u} = \sum_{n\geq 0}(-\partial_x)^n\frac{\partial}{\partial u_n}, \quad u_n = (\partial_x)^n u. \tag{46}$$

Using the Hamiltonian densities in (40) – (44), one can easily verify the Faddeev-Zakharov-Gardner equation in (45) to yield the appropriate nonlinear equations in (20) – (24). The bi-Hamiltonian form of evolution equations is given by [11]

$$u_t = \partial_x\left(\frac{\delta\mathscr{H}_{m+2}}{\delta u}\right) = \mathscr{E}\left(\frac{\delta\mathscr{H}_m}{\delta u}\right) \tag{47}$$

with $m = 2n+1, \quad n = 0,1,2,\ldots$. In (47) the second Hamiltonian operator is related to the recursion operator by [16]

$$\mathscr{E} = \Lambda\partial_x. \tag{48}$$

From (15) and (48) we get

$$\mathscr{E} = \partial_x^3 + \frac{2A}{5}u\partial_x + \frac{A}{5}u_x. \tag{49}$$

From (47) and (49) we have

$$\begin{aligned} u_t &= \partial_x \left(\frac{\delta \mathscr{H}_{m+2}}{\delta u} \right) \\ &= \left(\partial_x^3 + \frac{2A}{5} u \partial_x + \frac{A}{5} u_x \right) \left(\frac{\delta \mathscr{H}_m}{\delta u} \right). \end{aligned} \tag{50}$$

For $n = 1$, (50) reads

$$\begin{aligned} u_t &= \partial_x \left(\frac{\delta \mathscr{H}_5}{\delta u} \right) \\ &= \left(\partial_x^3 + \frac{2A}{5} u \partial_x + \frac{A}{5} u_x \right) \left(\frac{\delta \mathscr{H}_3}{\delta u} \right). \end{aligned} \tag{51}$$

From (41), (42) and (51) one can easily obtain (20) verifying the bi-Hamiltonian structure. Similar results can also be checked for other pairs of the Hamiltonians in (40) – (44).

4. Soliton Solution

We have just seen that the bi-Hamiltonian form (51) corresponds to the fifth-order nonlinear equation in (20). Here we shall make use of the homogeneous balance method (HB) [15] to construct an analytical expression for the soliton solution of this equation. According to the HB method, the field variable is first expanded as

$$u(x,t) = \sum_{i=0}^{N} f^{(i)}(w(x,t)), \tag{52}$$

where the superscript (i) denotes the derivative index. In particular, $f^{(1)} = \frac{\partial f}{\partial w}$, $f^{(2)} = \frac{\partial^2 f}{\partial w^2}$ and so on. Substituting (52) in (20) and balancing the contribution of the linear term with that of the nonlinear terms, the expression in (52) becomes restricted to

$$u(x,t) = f^{(2)} w_x^2 + f^{(1)} w_{2x}, \tag{53}$$

where the subscripts of w stand for appropriate partial derivatives. From (53) and (20) we have

$$\begin{aligned} &(f^{(7)} + A f^{(2)} f^{(5)} + 2A f^{(3)} f^{(4)} \\ &\quad + \frac{3A^2}{10} (f^{(2)})^2 f^{(3)}) w_x^7 \end{aligned} \tag{54}$$

+ other terms involving lower powers of the partial derivatives of $w = 0$. Setting the coefficient of w_x^7 to zero we get

$$\begin{aligned} &f^{(7)} + A f^{(2)} f^{(5)} + 2A f^{(3)} f^{(4)} \\ &\quad + \frac{3A^2}{10} (f^{(2)})^2 f^{(3)} = 0. \end{aligned} \tag{55}$$

If we try a solution of (55) in the form

$$f = \alpha \ln w, \tag{56}$$

we immediately get

$$\alpha = \frac{20}{A}. \tag{57}$$

From (56) we can deduce the following results:

$$\begin{aligned} &f^{(2)} f^{(5)} = -\frac{\alpha}{30} f^{(7)}, f^{(3)} f^{(4)} = -\frac{\alpha}{60} f^{(7)}, \\ &\left(f^{(2)}\right)^2 f^{(3)} = \frac{\alpha^2}{360} f^{(7)}, \\ &f^{(2)} f^{(4)} = -\frac{\alpha}{20} f^{(6)}, \left(f^{(3)}\right)^2 = -\frac{\alpha}{30} f^{(6)}, \\ &f^{(1)} f^{(5)} = -\frac{\alpha}{5} f^{(6)}, \\ &\left(f^{(2)}\right)^3 = \frac{\alpha^2}{120} f^{(6)}, f^{(1)} f^{(2)} f^{(3)} = \frac{\alpha^2}{60} f^{(6)}, \\ &f^{(2)} f^{(3)} = -\frac{\alpha}{12} f^{(5)}, f^{(1)} f^{(4)} = -\frac{\alpha}{4} f^{(5)}, \\ &\left(f^{(2)}\right)^2 f^{(1)} = \frac{\alpha^2}{24} f^{(5)}, \\ &\left(f^{(1)}\right)^2 f^{(3)} = \frac{\alpha^2}{12} f^{(5)}, \\ &f^{(1)} f^{(3)} = -\frac{\alpha}{3} f^{(4)}, \left(f^{(2)}\right)^2 = -\frac{\alpha}{6} f^{(4)}, \\ &\left(f^{(1)}\right)^2 f^{(2)} = \frac{\alpha^2}{6} f^{(4)}, \\ &f^{(1)} f^{(2)} = -\frac{\alpha}{2} f^{(3)}, \left(f^{(1)}\right)^3 = \frac{\alpha^2}{2} f^{(3)}, \\ &\left(f^{(1)}\right)^2 = -\alpha f^{(2)}. \end{aligned} \tag{58}$$

Substituting (58) in the full form of (54), the latter is reduced to a linear polynomial in $f^{(1)}$, $f^{(2)}, \ldots, f^{(7)}$. If the coefficient of each $f^{(i)}$ is set equal to zero we get a set of partial differential equations for $w(x,t)$:

$$w_{xxt} - w_{7x} = 0, \tag{59a}$$

$$\begin{aligned} &2 w_x w_{xt} + w_t w_{xt} + (2A\alpha - 35) w_{3x} w_{4x} \\ &\quad + (A\alpha - 21) w_{2x} w_{5x} - 7 w_x w_{6x} = 0, \end{aligned} \tag{59b}$$

$$\begin{aligned}&2w_t w_x^2+(A\alpha-42)w_x^2 w_{5x}\\&+(11A\alpha-210)w_x w_{2x}w_{4x}\\&+(8A\alpha-140)w_x w_{3x}^2\\&+\left(16A\alpha-\frac{3A^2}{10}\alpha^2-210\right)w_{2x}^2 w_{3x}=0,\end{aligned}\tag{59c}$$

$$\begin{aligned}&(48A\alpha-\frac{9A^2}{10}\alpha^2-630)w_x w_{2x}^3\\&+(78A\alpha-\frac{3A^2}{5}\alpha^2-1260)w_x^2 w_{2x}w_{3x}\\&+(9A\alpha-210)w_x^3 w_{4x}=0,\end{aligned}\tag{59d}$$

$$\begin{aligned}&\left(174A\alpha-\frac{12A^2}{5}\alpha^2-2520\right)w_x^3 w_{2x}^2\\&+\left(48A\alpha-\frac{3A^2}{10}\alpha^2-840\right)w_x^4 w_{3x}=0,\end{aligned}\tag{59e}$$

$$\left(24A\alpha-\frac{3A^2}{10}\alpha^2-360\right)w_x^5 w_{2x}=0\tag{59f}$$

and

$$\left(24A\alpha-\frac{3A^2}{10}\alpha^2-360\right)w_x^7=0.\tag{59g}$$

Equation (59a) is a linear partial differential equation and can be converted to an ordinary differential equation by substituting

$$w(x,t)=g(x+vt)=g(z).\tag{60}$$

Using (60) in (59a) we have

$$v\frac{\mathrm{d}^3 g}{\mathrm{d}z^3}-\frac{\mathrm{d}^7 g}{\mathrm{d}z^7}=0.\tag{61}$$

Here v is the velocity of the travelling wave represented by $w(x,t)$. Equation (61) can be solved to write

$$w(x,t)=g(x+vt)=c_0+c_1\mathrm{e}^{\sqrt[4]{v}(x+vt)},\tag{62}$$

where c_0 and c_1 are arbitrary constants. Using (56), (57) and (62) in (53) we get the exact soliton solution of the fifth-order equation in (20) and/or (51) in the form

$$u_5(x,t)=\frac{20}{A}\frac{c_0c_1\sqrt{v}\mathrm{e}^{\sqrt[4]{v}(x+vt)}}{(c_0+c_1\mathrm{e}^{\sqrt[4]{v}(x+vt)})^2}.\tag{63}$$

A similar result for the third-order equation in (22) is given by

$$u_3(x,t)=\frac{20}{A}\frac{c_0c_1v\mathrm{e}^{\sqrt{v}(x+vt)}}{(c_0+c_1\mathrm{e}^{\sqrt{v}(x+vt)})^2}.\tag{64}$$

The subscripts on $u(x,t)$ are self-explanatory. It is of interest to note that for $c_1=c_0=1$, $A=10$ and $v=4\kappa^2$, $u_3(x,t)$ in (64) becomes

$$u_3(x,t)=2\kappa^2\mathrm{sech}^2(\kappa x+4\kappa^3 t).\tag{65}$$

From the inverse spectral method [23] for solving the KdV equation, we know that κ^2 has a simple physical meaning. For example $-\kappa^2$ represents a discrete energy eigenvalue of the Schrödinger equation for the initial potential $u_3(x,0)$. As in [9] we shall now examine the spatial behaviour of $u_5(x,t)$ at $t=0$. For the sake of simplicity we shall work with $v=1$. In Fig. 1 we plot $u_5(x,0)$ as function of x for different values of the parameters c_0 and c_1. All the curves in the figure are of sech^2 shape, indicating that the solutions obtained from (63) have indeed solitary wave properties. The solid curve for $c_0=1$ and $c_1=1$ is centred at the point $x=0$. If c_0 and c_1 are made unequal, the centre of the soliton moves either to the left or to the right. In particular, for $c_0>c_1$, the shift of the centre is towards the right, and we have a reverse situation for $c_0<c_1$. We have displayed this property by using a dashed curve with cross ($c_0=4$ and $c_1=1$) and a simple dashed curve ($c_0=1$ and $c_1=4$).

5. Conclusion

Fifth-order nonlinear evolution equations, on the one hand have many connections with other important integrable equations and, on the other hand, can not be solved by simple analytical methods. These two points inspired us to construct a general fifth-order equation which follows from a Lagrangian. It is often desirable that equations of mathematical physics should be derivable from an action principle, because a non-Lagrangian system does not allow one to carry out a linear stability check [24] as well as to derive a field theory [25] for particles described by its solutions. The Lagrangian approach to nonlinear evolution equations is quite interesting because here one can derive all physico-mathematical results from first principles [8]. Based on the fifth-order Lagrangian equation we derived an integrable hierarchy. As a test of integrability we provided a Lax representation and constructed two compatible Hamiltoinan structures.

We treated the third- and fifth-order equations in the hierarchy by the homogeneous balance method [15] to obtain analytical results for soliton solutions. Ideally, we could have tried the bi-linear method of Hirota

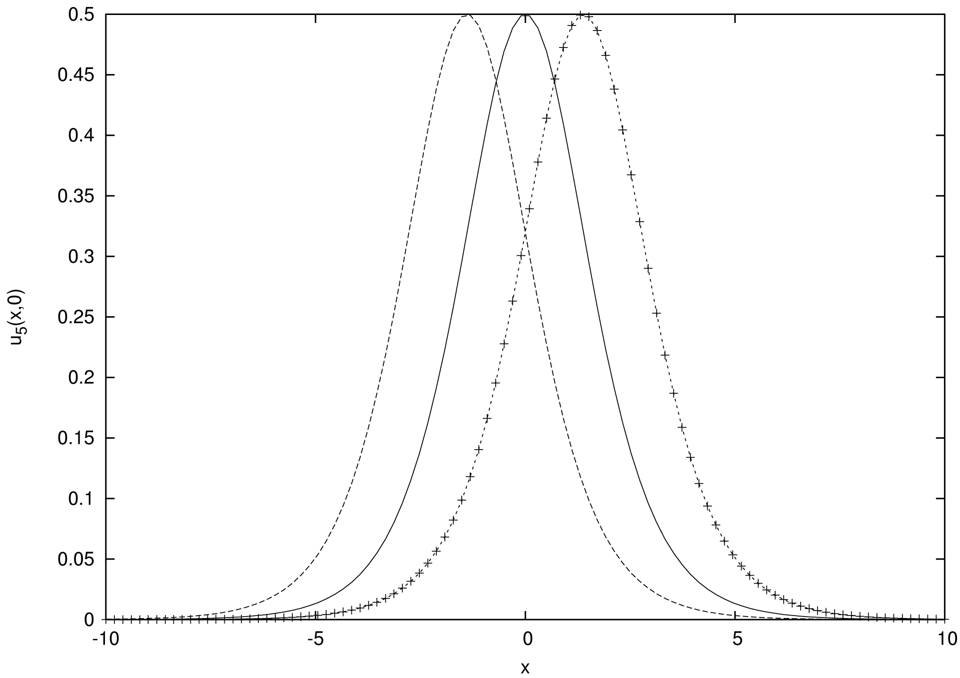

Fig. 1. Variation of $u_5(x,0)$ with x; $c_0 = c_1 = 1$: solid curve; $c_0 > c_1$: dashed curve with cross; $c_0 < c_1$: dashed curve.

[26] to deal with the problem because this method is very convenient for finding single- and multi-soliton solutions of nonlinear evolution equations. For higher-order equations, the Hirota transformation often leads to a multilinear representation [27]. This tends to pose problems in solving the equations. The homogeneous balance method, on the other hand, does not involve any new mathematical complication as one moves from lower- to higher-order equations. Admittedly, the algebra becomes more and more involved as we go up the ladder inside the hierarchy. The symbolic computations like Maple and Mathematica can be used to circumvent algebraic complications.

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